Time-optimal control for pedestrians: algebraic SU(2) generation with alternating controls

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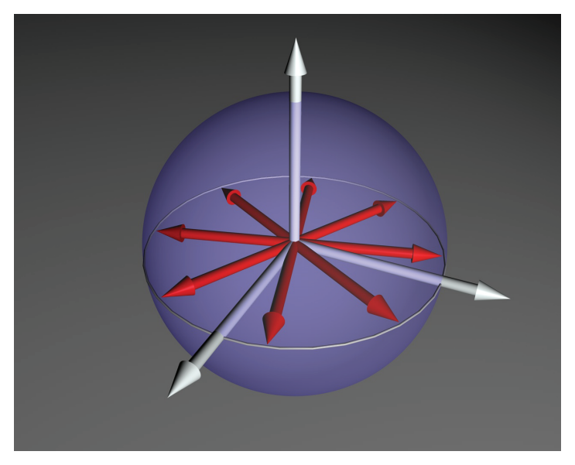
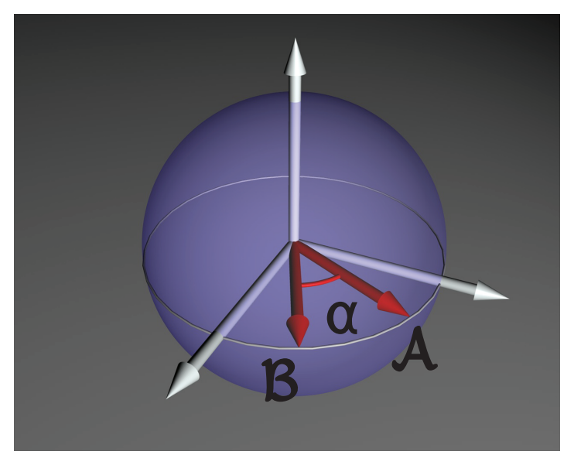
Time-optimal control with algebraic methods: the case of alternating controls

Time-optimal control theory studies the task of steering a dynamical system towards some goal state in minumum time, while obeying some constraints: $\dot{X}(t) = AX(t) + B\underbrace{u(t)}_{\text{control}}$, with $|u(t)| \leqslant 1$.

In NMR-like systems, phase/amplitude modulation of the excitation fields allows u(t) to vary in [-1,1]. Nevertheless, in some experimental settings, it is either hard to modulate the fields (ex: ion traps with laser controls), or the control degrees of freedom are restricted (ex: evolution of nuclear spin coupled to the electronic NV-center spin: the driving of the NV between the $m_s = \{1,0\}$ states turns hyperfine on/off). In these cases, the control is constraint to alternate: $u(t) = \pm 1$.



$u(t) \in [-1,1]$



$u(t) = \pm 1$

$\alpha \in [0,\pi]$
$AX(t) + B \equiv \mathcal{A}$
$AX(t) - B \equiv \mathcal{B}$

The standard approach to time-optimal control usually involves geometrical and variational methods. Such methods either fail or become convoluted in the case of alternating controls.

Using accessible algebraic methods, we obtain general results for the time-optimal generation of unitaries in SU(2) using alternating controls from a set $\{\mathcal{A}, \mathcal{B}\}$ that generates the whole space:

$$U_{goal} = \mathcal{A}(t_n)\, \mathcal{B}(t_{n-1})\, \mathcal{A}(t_{n-2}) \ldots \mathcal{B}(t_3)\, \mathcal{A}(t_2)\, \mathcal{A}(t_1)$$

The evolution of the system towards U_{goal} is then understood as consecutive rotations around two non-parallel axes in the Bloch sphere. Our task is to determine the length and the times of the time-optimal decomposition of U_{goal}. Here, time-optimality is the only criterium for optimization, although others exist (ex: minimum number of control pulses).

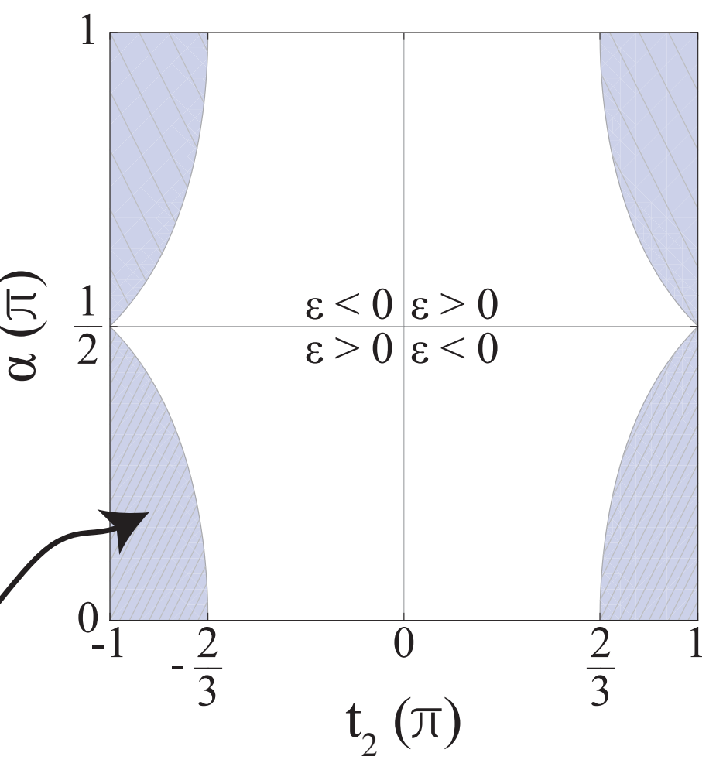
Toolbox and tricks

We use a myriad of algebraic tricks to find the time-optimal decomposition of a given unitary, some of which are below:

Extend the number of controls in a sequence, while reducing the total time:
it is always possible to find

$$U_{goal} = \mathcal{A}(t_3)\, \mathcal{B}(t_2)\, \mathcal{A}(t_1)$$
$$= \mathcal{A}(t_3\text{-}\varepsilon)\, \mathcal{B}(\tau)\, \mathcal{A}(\mu)\, \mathcal{B}(\tau)\, \mathcal{A}(t_1\text{-}\varepsilon)\ ;$$

total times shorter in the blue region



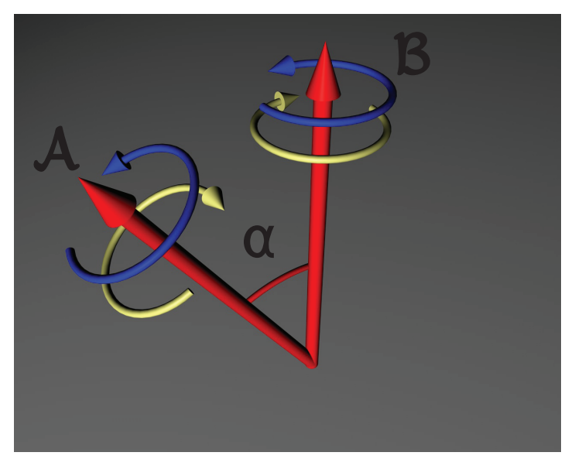
Alternative decompositions of the same unitary:

$U_{goal} = \mathcal{A}(t_3)\, \mathcal{B}(t_2)\, \mathcal{A}(t_1) = \mathcal{A}(t_3\text{–}t^*)\, \mathcal{B}(2\pi\text{–}t_2)\, \mathcal{A}(t_1\text{–}t^*)$,
with $t^* = 2\ \text{arccot}(\cos(\alpha)\ \tan(t_2/2))$

Variational principle for time-optimal sequences:

$U_{goal} = \mathcal{A}(t_n) \ldots \mathcal{B}(t_3)\, \mathcal{A}(t_2)\, \mathcal{B}(t_1)$ is time optimal iff
$U_{goal} + dU_{goal} \approx U_{goal} = \mathcal{A}(t_n{+}\varepsilon_n) \ldots \mathcal{A}(t_2{+}\varepsilon_2)\, \mathcal{B}(t_1{+}\varepsilon_1)$,
for small perturbations ε_i; yields the important result that internal times must be equal in magnitude [1]:
$|t_2| = |t_3| = \ldots = |t_{n-1}| \equiv |t_m|$

Case $t \in [-\pi,\pi]$: shape of time-optimal solution



Not a lot [2] was known in the case where rotations can be made in both clock- and anti-clockwise directions, besides the fact that all times in the time-optimal decomposition must be $t \in [-\pi,\pi]$.
We now know that, if the globally time-optimal (*gto*) sequence is finite, the signs of the middle times follow:

$$\begin{aligned}
&\ldots \mathcal{A}(+t_m)\ \ \mathcal{B}(+t_m)\ \mathcal{A}(-t_m)\ \ \mathcal{B}(-t_m) \ldots \\
&\ldots \mathcal{A}(-t_m)\ \ \mathcal{B}(-t_m)\ \mathcal{A}(+t_m)\ \ \mathcal{B}(+t_m) \ldots \\
&\ldots \mathcal{A}(+t_m)\ \ \mathcal{B}(-t_m)\ \mathcal{A}(-t_m)\ \ \mathcal{B}(+t_m) \ldots \\
&\ldots \mathcal{A}(-t_m)\ \ \mathcal{B}(+t_m)\ \mathcal{A}(+t_m)\ \ \mathcal{B}(-t_m) \ldots
\end{aligned}$$

Conversely, if the *gto* sequence is infinite, $t_m \rightarrow 0$, with signs:

$$\begin{aligned}
&\ldots \mathcal{A}(+t_m)\ \ \mathcal{B}(+t_m)\ \mathcal{A}(+t_m)\ \ \mathcal{B}(+t_m) \ldots \\
&\ldots \mathcal{A}(-t_m)\ \ \mathcal{B}(-t_m)\ \mathcal{A}(-t_m)\ \ \mathcal{B}(-t_m) \ldots \\
&\ldots \mathcal{A}(+t_m)\ \ \mathcal{B}(-t_m)\ \mathcal{A}(+t_m)\ \ \mathcal{B}(-t_m) \ldots \\
&\ldots \mathcal{A}(-t_m)\ \ \mathcal{B}(+t_m)\ \mathcal{A}(-t_m)\ \ \mathcal{B}(+t_m) \ldots
\end{aligned}$$

Case $t \in [0,2\pi]$: bounds on length and times of time-optimal decomposition

If the *gto* for U_{goal} is finite, then $t_m > \pi$ [1].

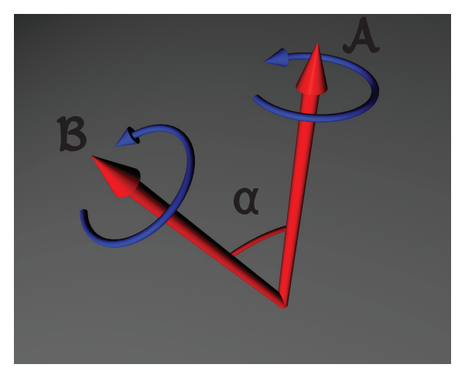
We have additionally concluded that:

✓Given any U_{goal} in SU(2), we can determine the minimum number of control concatenations, κ, necessary to synthesize U_{goal}.

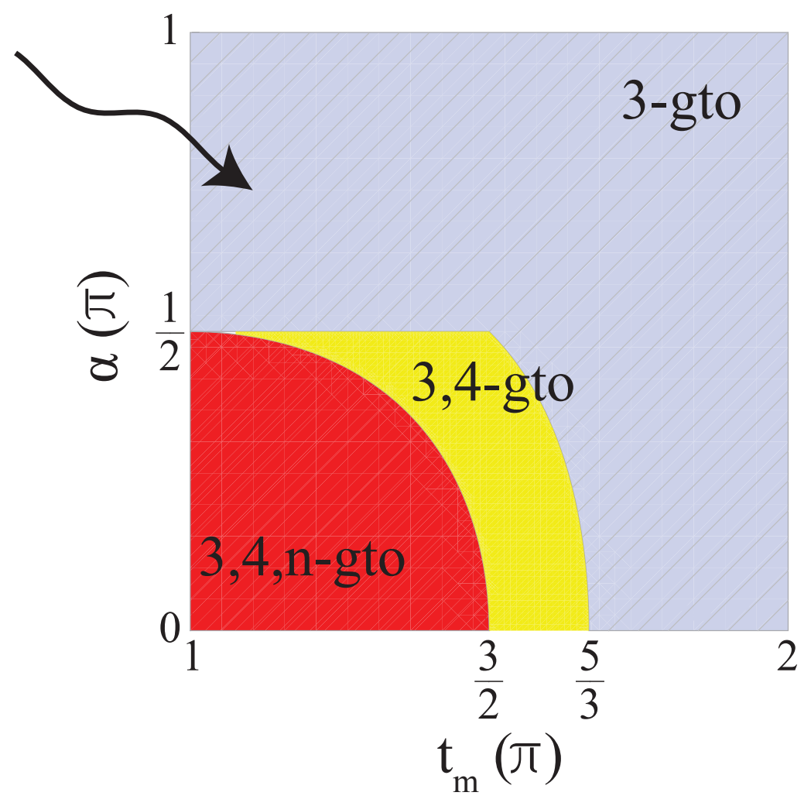
✓If the best n-long decomposition of U_{goal}, $n \geqslant \kappa$, has $t_m < \pi$ (ie, is *not* a *gto*-candidate), the *gto* must be (n+1), (n+3) or infinitely long.

✓If the best n-long decomposition of U_{goal}, $n \geqslant \kappa$, has $t_m > \pi$ (ie, is a *gto*-candidate), the gto must be n, (n+1), (n+2), . . . , $\lceil(3n/2)+3\rceil$ or infinitely long. (Conjecture: shortest *gto*-candidate is indeed *gto*!)

✓If an infinite decomposition of U_{goal} exists, a sequence n-long, with $n \geqslant (6 + 2/\cos(\alpha/2))$, can never be *gto* (infinite sequence performs better).



✓If the *gto* is finite, we can bound the length of the time-optimal sequence according to the relationship between t_m and α. Note that, for $\alpha \geq \pi/2$, the *gto* is either 3 or infinitely long!



Numerical simulations indicate that the time improvement brought by infinite *gto*-sequences is usually very small, and we are currently analytically investigating this quantity.

Also of current interest is the generalization of our formalism for different rotation speeds.

[1] Y. Billig, Quantum Inf. Process. 12, 955 (2013)
[2] U. Boscain, H. Rabitz et al., arXiv 1211.0666